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OPTIMIZATION OF THE SHAPES OF OBSTACLES IN JET- SEPARATION FLOW

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The model of an ideal incompressible fluid is used to study the solvability of optimal control problems for the shape of a nozzle which discharges free-boundary fluid flow with and without accounting for gravity (internal aerodynamics) and shape optimization problems for an obstacle with jet separation (external aerodynamics). The qualitative properties of such flows are studied.

Key words: *conformal mappings, free boundary, cavitation, optimal control, body shape.*

In 1935, Lavrent'ev, using the variational principles of conformal mappings he developed, showed that for ideal-fluid jet flow over convex arcs, a circular arc has the maximum lift force [1, pp. 405–449]. The optimal control of solutions of elliptic equations for a wide range of target functionals is considered in [2].

Problems of controlling the nozzle shape or the shape of an obstacle in supersonic fluid flow have been the subject of extensive research [3, 4]). Such problems allow one to use analogs of the Pontryagin principle and thus to design algorithms of their numerical solution.

1. JET FLUID FLOW FROM AN OPTIMAL NOZZLE

1.1. Formulation of the Problem. Let an incompressible fluid discharge from a polygonal nozzle $P_0 = (z_0, \ldots, z_{n-1})$ with vertices $z_k = x_k + iy_k$ and vertex angles $\alpha_k\pi$ onto an infinite rectilinear confining layer $P_1 = (z_n, z_{n+1})$ ($y_{n+1} = 0$, $x_n = -\infty$, and $x_{n+1} = \infty$). In this case, the segment $P_2 = (z_{n-1}, z_n)$ is a horizontal straight line ($y_{n-1} = y_n = H$) (see Fig. 1). In the domain D bounded by the polygon $P = (z_0, \ldots, z_{n+1}) = P_0 \cup P_1 \cup P_2$ and the unknown curve (jet) $L = (z_{n+1}, z_0)$, $\partial D = P \cup L$, we seek a complex flow potential $w(z) = \varphi + i\psi$ (an analytic function of the variable $z = x + iy$) that satisfies the boundary conditions

$$\psi = 0, \quad z \in P_1, \qquad \psi = Q, \quad z \in P_0 \cup L, \qquad \Big|\frac{dw}{dz}\Big| = 1, \quad z \in L, \tag{1}$$

where $Q = \mathrm{const} > 0$ is the required flow discharge.

The derivatives of the conformal mappings $w\colon\ K \to D^*$ and $z\colon\ K \to D$ of the unit semicircle $K = \{\zeta\colon |\zeta| < 1,\ \operatorname{Im}\zeta > 0\}$ onto the strip $D^* = \{w\colon\ 0 < \operatorname{Im} w < Q\}$ and the domain D, respectively, are represented, as follows [5, p. 178]:

$$\frac{dw}{d\zeta} = N_0(1-\zeta^2)\prod_{k=n}^{n+1}[(\zeta - t_k)(1 - \zeta t_k)]^{-1} \equiv N_0\omega(\zeta), \qquad \frac{dz}{d\zeta} = N_0\omega(\zeta)\prod_{k=1}^{n-1}\Big(\frac{\zeta - t_k}{1 - \zeta t_k}\Big)^{\beta_k}. \tag{2}$$

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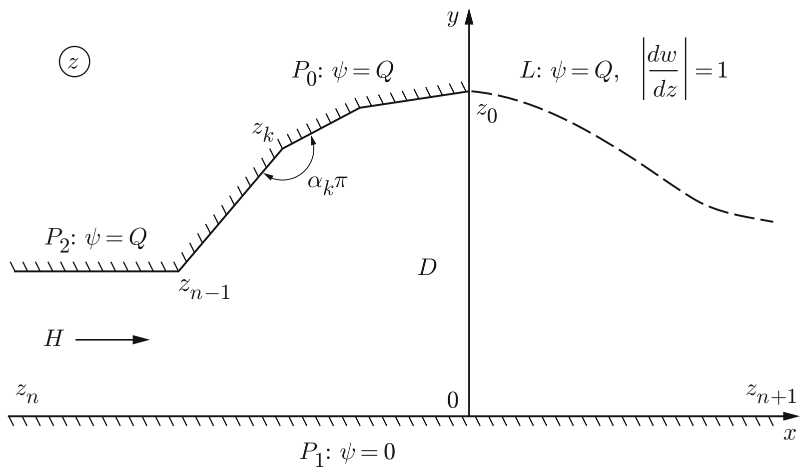

Fig. 1. Diagram of jet fluid flow from the nozzle.

Here t_k ($t_0 = -1 < t_1 < \ldots < t_n < t_{n+1} = 1$) are the preimages of the vertices z_k of the polygon P under the conformal mapping $z = z(\zeta)$, $\beta_k\pi = (\alpha_k - 1)\pi$ are the external angles for z_k, and $N_0(Q) = \text{const} > 0$.

1.2. Equations for the Parameters. We fix the points $z_0 = ih_0$ and $z_{n-1} = x_{n-1} + iH$ [h_0, H, and x_{n-1} ($-\infty < x_{n-1} < 0$) are known constants].

For the polygons $P_0 = (z_0, \ldots, z_{n-1})$, we introduce *the geometric characteristic* $p_0 = (l, \beta)$, $l = (l_1, \ldots, l_{n-1})$, $l_k = |z_k - z_{k-1}|$, and subject it to the *simple polygon* conditions $(p_0, P_0) \in G(\delta)$:

$$G: \qquad \delta - 1 \le \beta_k \le 1, \quad |\ln l_k| \le \delta^{-1}, \quad k = \overline{1, n-1} \quad (0 < \delta \ll 1). \tag{3}$$

The parameters N_0 and t_n in (2) are specified, and the constants t_k corresponding to the finite tips $z_k \in P_0$ ($k = \overline{1, n-1}$) are found as solutions of the nonlinear system of equations [5, p. 162]:

$$l_k = \int\limits_{t_{k-1}}^{t_k} \left|\frac{dz}{dt}\right| dt \equiv g_k(T, \beta), \qquad k = \overline{1, n-1}.$$

Here $T = (t_1, \ldots, t_{n-1})$, $\beta = (\beta_1, \ldots, \beta_{n-1})$, and $\beta_k\pi = (\alpha_k - 1)\pi$ are the external angles for $z_k \in P_0$.

The solvability of the system of equations for T was established in [5] using the continuity method with the inclusion $T \subset R$ being proved:

$$R: \qquad t_{k+1} - t_k > \varepsilon(\delta) > 0, \quad k = \overline{0, n}. \tag{4}$$

The system of equations for t_k is written as one functional equation

$$l = g(T, \beta), \qquad (l, \beta) \in G, \qquad g = (g_1, \ldots, g_{n-1}). \tag{5}$$

With the satisfaction of the additional constraint

$$0 < \delta \le |\theta(t) - \beta| \le \pi - \delta, \qquad \theta = \arg\frac{dz}{dt}, \qquad |t| \le 1 \tag{6}$$

on the rotation of the tangent to the polygon P_0 (β is a certain angle) for the Kirchhoff model [5, pp. 153–156] and Ryabushinskii model [6], it was proved that the solutions T of Eq. (5) are locally unique:

$$\left|\frac{Dg}{DT}\right| \ge \varepsilon_0(\delta) > 0. \tag{7}$$

Here $Dg/DT = \{\partial g_i/\partial t_j\}$ is a Jacobi matrix. It was established in this case that if among the polygons P_0 there is a polygon P_0^0 for which the solution of Eq. (5) is unique, it is also unique for any finite polygon $P_0 \in G$. In this case, as P_0^0 one can use a straight-wall nozzle $P_0^0 = \{y = y_0,\ -\infty < x < x_0\}$, for which Eq. (5) is satisfied automatically for any partition of P_0^0 by the points $z_k = x_k + iy_0$, $k = \overline{1, n-1}$, $|z_{k+1} - z_k| \ne 0, \infty$. The validity of the estimate (7) for the general cavitation model including the problem considered, is established in Sec. 3.

1.3. Optimization Problem. We determine the geometrical characteristic p_0 of the required polygon P_0 from a certain optimality condition. In problems of internal aerodynamics, the target functional is usually the thrust at the optimized-nozzle exit.

We draw a semicircle $K_0 = \{\zeta\ |\zeta + \varepsilon/4| < r_0 = 1 - \varepsilon/4,\ \operatorname{Im}\zeta > 0\}$ through the points $t_0 = -1$ and $t_* = (1-\varepsilon/2) \in (t_n, t_{n+1})$, and denote by $\Gamma_0 = \{r_0\, \mathrm{e}^{i\gamma}\colon\ 0 < \gamma < \pi\}$ the preimage of the curve connecting the points $z_0 \in P_0$ and $z_* = z(t_*) \in P_1$. The thrust functional is chosen in the form

$$F = \int\limits_0^\pi \Big|\frac{dw}{dz}(\zeta)\Big|^2\, d\gamma = C\int\limits_0^\pi \prod_{k=1}^{n-1}\Big|\frac{1-\zeta t_k}{\zeta - t_k}\Big|^{2\beta_k}\, d\gamma \qquad (\zeta = r_0\, \mathrm{e}^{i\gamma}). \tag{8}$$

The functional $F(P_0)$ has bounded derivatives of any finite order with respect to the arguments (β_k, t_j), which enter explicitly because the integrand in (8) has no singularities (the points t_0 and t_{n+1} do not enter the product). According to [5], the solutions $t_j = t_j(l, \beta)$ are also differentiable with respect to (l_i, β_k). Therefore, the functional $F(P_0)$ has an extreme point P_0^* [7, p. 106]:

$$\sup_G F(P_0) = F(P_0^*). \tag{9}$$

We start the optimization for the case of a curved nozzle with increasing the number of the vertices of the polygon P_0 by introducing the notation $P_0^m = (z^0, \dots, z^m)$ ($z^0 = z_0$; $z^m = z_{n-1}$).

We seek the nozzle $\Lambda \subset C^1$ in the class of piecewise smooth curves of finite length:

$$C^1\colon \qquad \Big|\ln\frac{dz}{d\tau}\Big| \le M, \quad |z(\tau)| \le M, \quad \tau \in [0,1].$$

Here $z = z(\tau)$ is the parametric equation of Λ. To each given curve Λ, we assign the set of polygons converging to it $P^m \to \Lambda$. Then, as shown in [5, p. 168–170], the set of conformal mappings $Z^m\colon K \to D$ that corresponds to P_0^m is uniformly bounded in the domain $K_\delta \equiv K \setminus Q_\delta(t_n, t_{n+1})$ [Q_δ is a fixed δ-neighborhood ($0 < \delta \ll 1$) of the given points t_n and $t_{n+1} = 1$]. This allows us to distinguish a convergent subsequence $\{Z^{m_k}(\zeta)\}$, $Z^{m_k}(\zeta) \to Z(\zeta)$ for $m_k \to \infty$, and the limit mapping $Z\colon K_\delta \to D_\delta$ transforms the segment $[-1, t_{n-1}]$ to the curve Λ. The behavior of the mapping $z = Z(\zeta)$ in the neighborhood of Q_δ is also described in [5].

The properties of the thrust functional $F(P_0^{m_k})$ allow us to pass to the limit as $m_k \to \infty$ and to find the limiting optimal curved nozzle $\Lambda^* \subset C^1$:

$$\lim_{m_k\to\infty} \sup_G F(P_0^{m_k}) = F(\Lambda^*). \tag{10}$$

Theorem 1. *On the set of simple finite polygons $P_0 \in G$ there exists an extreme point P_0^* of the functional $F(P_0)$, i.e., equality* (9) *is satisfied. If condition* (6) *is satisfied, each extreme point of the functional $F(P_0)$ is isolated.*

In the class of curves $\Lambda \subset C^1$, $\Lambda = \lim\limits_{m_k\to\infty} P_0^{m_k}$, $P_0^{m_k} \in G$ of finite length $|\Lambda| \le M$, $M \ge |z_{n-1} - z_0|$ there exists an optimal curved nozzle Λ^* that satisfies relation (10).

As noted above, if condition (6) is satisfied, the solution $T(P_0)$ $\forall P_0 \in G$ of Eq. (5) is unique, and, hence, it is also unique for the extreme point P_0^* of the functional $F(P_0)$, which implies that this point is insulated.

2. ACCOUNTING FOR GRAVITY

In the problem considered above, the condition $q \equiv |dw/dz| = 1$ on the free boundary L is replaced by Bernoulli's equation

$$q^2 + 2gy = q_\infty^2 + 2gh, \tag{11}$$

where g is the acceleration of gravity, q_∞ and h are the flow velocity and depth, respectively, at infinity downstream. This problem was first studied by Monakhov (1969) without invoking any smallness conditions on the flow parameters [5, pp. 178–184]. The method used to solve this problem consisted of simultaneous approximation of curved boundaries by polygons and linearization of boundary condition (11) in such a manner that it was satisfied at a finite number of points on L. After the solvability of the thus obtained auxiliary problems was proved, the solvability of the initial problem was established by passing to the limit.

The scheme for solving the problem of heavy fluid flows is briefly described below.

2.1. Auxiliary Problem. Let the parameters $a_0 = q_0 q_\infty^{-1}$, $q_0 = |dw/dz|_{z=z_0}$, and $\mu = h/h_0$ ($h_0 = \operatorname{Im} z_0$) be specified. The quantities q_0, q_∞, h, h_0, and Q are sought together with the complex flow potential $w = w(z)$. The constants a_0 and μ obey the natural constraints for any value of $\delta \ll 1$:

$$0 < \delta \le \mu = h h_0^{-1} \le 1 - \delta, \qquad \mathrm{e}^{-2\pi} - \delta \le a_0 = q_0 q_\infty^{-1} \le 1 - \delta. \tag{12}$$

The representation for $dw/d\zeta$ from Sec. 1 is retained, and representation (2) is replaced by the following:

$$\frac{dz}{d\zeta} = N_0 \omega(\zeta) \prod_{k=1}^{n-1} \Big(\frac{\zeta - t_k}{1 - \zeta t_k}\Big)^{\beta_k} \mathrm{e}^{M(\zeta)} . \tag{13}$$

Here

$$M = \frac{\zeta^2 - 1}{\pi} \int_0^\pi \ln\,[q_\infty^{-1} q(\gamma)]\,\frac{d\gamma}{1 - 2\zeta\cos\gamma + \zeta^2},$$

$$[q_\infty^{-1} q(\gamma)]^2 = 1 + 2g q_\infty^{-2} [h - y(\gamma)],$$

where $y = y(\gamma)$ is the required function [5, p. 180].

We partition the sought interval $[h, h_0]$ by the points

$$y^k = h_0 - k\,\frac{h_0 - h}{m+1} \qquad (k = \overline{0, m+1}) \tag{14}$$

and set $z^k = (x^k + iy^k) \in L$. Let $\zeta_k = \mathrm{e}^{i\gamma_k}$ be the preimages of z^k and $q_k = q(\gamma_k)$ and $q_{m+1} = q_\infty$. From equalities (11) and (14), we obtain

$$q_{k+1}^2 - q_k^2 = 2gh\,\frac{1-\mu}{\mu(m+1)}, \qquad k = \overline{0, m}. \tag{15}$$

We introduce the functions

$$\tilde q_k(\gamma) = \exp\Big(p_\infty^{k+1} + \frac{\cos\gamma - \cos\gamma_{k+1}}{\cos\gamma_k - \cos\gamma_{k+1}}\, p_{k+1}^k\Big), \qquad \gamma \in [\gamma_{k+1}, \gamma_k] \tag{16}$$

and in (13) we replace $q(\gamma)$ by the quantity $\tilde q(\gamma) = \tilde q_k(\gamma)$, $\gamma \in [\gamma_{k+1}, \gamma_k]$. Then, by the construction, Bernoulli's equation (11) is satisfied at a finite number of points $z^k \in L$, $k = \overline{0, m+1}$. To find the unknowns γ_k, we have the system of equations

$$\frac{y^k - y^{k-1}}{h} = \frac{(1 - t_n)^2}{\pi} \int_{\gamma_{k+1}}^{\gamma_k} \frac{(1+\zeta)\sin\theta(\gamma)\,d\gamma}{|1-\zeta|\,|\zeta - t_n|^2 \tilde q_k(\gamma)}, \qquad k = \overline{0, m-1}, \tag{17}$$

where $\theta(\gamma) = \arg\,(d\tilde z/d\gamma)$; $d\tilde z/d\gamma$ is defined by formula (13), in which $q(\gamma)$ is replaced by $\tilde q(\gamma)$.

2.2. A Prior Estimates. As in Sec. 1, the constants N_0 and t_n are fixed, and to determine the required parameters t_k ($k = \overline{1, n-1}$) and γ_k ($k = \overline{1, m}$) in the auxiliary problem, we obtain system (5), (11) with the total number of equations $m + n + 1$ [in Eqs. (5), dz/dt is replaced by $d\tilde z/dt$]. In this case, estimates (4) hold true for the vector $T = (t_1, \ldots, t_{n-1})$. We prove the same inclusion for the vector $(\gamma_1, \ldots, \gamma_m) \in R_\gamma$:

$$R_\gamma\colon \qquad \gamma_i - \gamma_{i+1} > \varepsilon_0 > 0, \quad i = \overline{0, m}.$$

Because of the boundedness of $|z_0|$ and $y^k = \operatorname{Im} z^k$, the convergence $|z_0 - z^k| \to 0$ is possible only as $\operatorname{Re} z^k = x^k \to \infty$. Then, the flow depth at infinity downstream is equal to $h^* = y_0 - y^k \neq h$, which contradicts the initial assumption. Starting from some γ_p ($1 \le p \le m$), let $\gamma_k \to 0$ ($k \ge p$). Then,

$$|z^p - z^{p-1}| = \frac{(1-t_n)^2}{\pi}\Big| \int_{\gamma_p}^{\gamma_{p-1}} \frac{|1+\zeta|\,\mathrm{e}^{i\theta(\gamma)}\;d\gamma}{|1-\zeta|\,|\zeta - t_n|^2 \tilde q_{p-1}(\gamma)}\Big| \equiv \Big| \int_{\gamma_p}^{\gamma_{p-1}} \frac{f(\gamma)\,d\gamma}{\sin\gamma/2}\Big|,$$

and, since $|f(\gamma)| > 0$ in the neighborhood $\gamma = 0$, the integral in the last equality diverges as $\gamma_p \to 0$, i.e., $|z^p - z^{p-1}| \to \infty$ as $\gamma_p \to 0$.

Thus, it is proved that $|\gamma_k| > \varepsilon_0 > 0$ $(k = \overline{0,m})$. At the same time, if $\gamma_k - \gamma_{k+1} \to 0$, from system (17), we obtain

$$\frac{1-\mu}{\mu(m+1)} = \frac{(1-t_n)^2}{\pi} \int\limits_{\gamma_{k+1}}^{\gamma_k} \frac{|1+\zeta| \sin\theta(\gamma)\, d\gamma}{|1-\zeta|\, |\zeta - t_n|^2 \tilde{q}_k(\gamma)} \le \frac{\gamma_k - \gamma_{k+1}}{a_0 \sin(\gamma_{k+1}/2)} \to 0,$$

which contradicts the condition $\mu < 1$. Therefore, there exists $\varepsilon > 0$ such that $\gamma_k - \gamma_{k+1} > \varepsilon > 0$ $(k = \overline{0,m})$. The latter circumstance provides for the satisfaction of the estimates

$$c_0^{-1} \le |\,\mathrm{e}^{M(\zeta)}\,| \le c_0 \quad \text{at} \quad \zeta \in D_\zeta = \{|\zeta| \le 1,\ \operatorname{Im}\zeta \ge 0\}.$$

2.3. Solvability of the Problem. We consider the simple problem of fluid discharge from a straight-wall nozzle $P^0 = (z_0, z_n)$, requiring that Bernoulli's equation be satisfied only at the jet discharge point z_0 and at the point z_{n+1} at infinity downstream.

The solvability of system (5), (17) corresponding to P^0 is easily established [5, p. 183].

In the equation of system (5), we introduce the parameters λ such that the polygon $P = (z_0, \ldots, z_n)$ becomes P^0 as λ varies from unity to zero. In addition, we include the parameter λ in the boundary condition by setting

$$q^\lambda(\gamma) = \lambda \tilde{q}(\gamma) + (1-\lambda) q^0(\gamma).$$

Here $\tilde{q}(\gamma)$ corresponds to the problem for P, and $q^0(\gamma) = \exp\{[(1-\cos\gamma)/2] \ln(q_0/q_\infty)\}$ to the problem for P^0. Because the estimates $dz^\lambda/d\zeta$ corresponding to the polygons P^λ are uniform with respect to λ, we use the Leray–Schauder fixed point theorem to establish the solvability of the corresponding system (5), (17) for any $\lambda \in [0,1]$.

Now, for an arbitrary simple polygonal nozzle $P_0 = (z_0, \ldots, z_{n-1})$, we can pass to the limit as $m \to \infty$, because of which Bernoulli's equation (11) is satisfied on the entire free boundary L [5, p. 184].

2.4. Optimal Polygonal Nozzle. We seek $P_0 = (z_0, \ldots, z_{n-1})$ from the maximum condition for the thrust functional $F(P_0)$ specified by formula (8), in which the factor $\mathrm{e}^{-2\operatorname{Re} M(\zeta)}$ needs to be added in the integrand. These changes in the form of the thrust functional $F(P_0)$ do not influence its differential properties, which (as in Sec. 1) allows us to find an extreme point P_0^* that satisfies equality (9). By passing to the limit from the polygons to the curve $\Lambda \subset C^1$ $(P_0^{m_k} \to \Lambda)$, we find the optimal curved nozzle Λ^* that satisfies relation (10).

Theorem 2. *In the problem of heavy-fluid discharge from a polygon nozzle $P_0 \in G(\delta)$ there exists an extreme point P_0^* of the thrust functional $F(P_0)$ that satisfies relation* (9). *By passing to the limit from the polygons to the curved boundaries, we also find the optimal curved nozzle $\Lambda^* \subset C^1$ that satisfies equality* (10).

3. OPTIMAL CONTROL OF CAVITATION

In applied hydrodynamics, the drag of bluff bodies is reduced using artificial cavitation methods. Cavitators are produced so that the body is inside the cavern formed behind the cavitator; air is frequently injected into the cavern; special devices produce flow inside the cavern to equalize pressure, etc. This leads to the implementation of various well-known cavitation models: Kirchhoff, Ryabushinskii, Efros, Efros–Joukowski, Joukowski–Roshko, etc. [5, p. 174–178].

Important applied problems are problems of optimal control of cavitation (sizes or drag of the cavern) by varying the cavitator shape and by pressure redistribution inside the cavern.

3.1. Jet Problem. We consider the general free-boundary hydrodynamic problem formulated and studied in [5, Ch. 4, Sec. 2], which includes all basic cavitation models. Let a domain D be bounded by a free surface L on which $|dw/dz| = 1$, and by a simple polygon $P = (z_0, \ldots, z_{n+1})$. In this case, the domain $\overline{D}$ can contain the flow stop and bifurcation points A_i and C_m $[(dw/dz)(A_i) = (dw/dz)(C_m) = 0]$ and the points B_j at which the vortices and sources $[(dw/dz)(B_j) = \infty]$ are located. Then, the derivatives of the conformal mappings of the upper half-plane $E = \{\zeta\colon \operatorname{Im}\zeta > 0\}$ in the domain D and D^* are written as [5, 8]

$$\frac{dw}{d\zeta} = N_0 \omega(\zeta), \qquad \omega = \prod_{i,j} \frac{\zeta^2 - |a_i|^2}{\zeta^2 - |b_j|^2} \prod_{m,s} \frac{\zeta - c_m}{\zeta - \sigma_s},$$

$$\frac{dz}{d\zeta} = N_0\omega(\zeta)\chi^{\nu}(\zeta)\Pi(\zeta)\,\mathrm{e}^{M(\zeta)}, \qquad \Pi = \prod_k \chi_k^{\beta_k}. \tag{18}$$

Here a_i, b_j, and c_m are the specified preimages of A_i, B_j, and C_m, respectively, σ_s are the fixed preimages of the infinite vertices $w(\sigma_s) \in \partial D^*$,

$$\chi_k = [(1-\zeta^2)^{1/2}(1-t_k^2)^{1/2} + 1 - \zeta t_k](\zeta - t_k)^{-1}, \qquad \chi = (1-\zeta^2)^{1/2} + 1,$$

$$M = -\frac{(1-\zeta^2)^{1/2}}{\pi i} \int\limits_{|t|>1} \frac{\ln |\Pi(t)\chi^{\nu}(t)|\, dt}{(1-t^2)^{1/2}(t-\zeta)},$$

$t_k \in (-1, 1)$ are the required preimages of the finite vertices P ($t_0 = -1$ and $t_{n+1} = 1$), and ν is an integer. The parameters N_0 and t_n are specified, and the vector $T = (t_1, \ldots, t_{n-1})$ is determined from Eq. (5), whose solution satisfies estimates (4) [5].

By the construction, the function $dz/d\zeta$ in the form (18) satisfies the boundary-value problem

$$\arg \frac{dz}{dt} = \pi\bar{\theta}(t), \quad |t| < 1, \qquad \left|\frac{dz}{dt}\right| = |N_0\omega(t)|, \quad |t| > 1, \tag{19}$$

where $\bar{\theta}(t) = \bar{\delta}_k$, $t \in [t_k, t_{k+1}]$; $\bar{\delta}_k\pi$ is the slope of the kth side of the polygon P to the Ox axis.

3.2. Local Uniqueness of the Solutions. According to the continuity method, to prove the uniqueness of the solution $T = (t_1, \ldots, t_{n-1})$ of Eq. (5) in the general jet problem (see Subsec. 3.1), it is sufficient to establish that its Jacobian Dl/DT differs from zero because in the simple case where the polygon P is a segment of the straight line, the uniqueness of the solution is known [5].

We associate the general jet problem with an auxiliary Kirchhoff flow, assuming that

$$\frac{dW}{d\zeta} = N_1\zeta\Big(\equiv Q_0(\zeta)\,\frac{dw}{d\zeta}\Big), \qquad \frac{dZ}{d\zeta} = N_2\chi(\zeta)\Pi(\zeta)\Big(\equiv Q(\zeta)\,\frac{dz}{d\zeta}\Big). \tag{20}$$

Here $W\colon\ E \to \Omega^*$ and $Z\colon\ E \to \Omega$ are conformal mappings that correspond to the Kirchhoff problem (20) in some domains Ω and Ω^*; the derivatives $dw/d\zeta$ and $dz/d\zeta$ are given by formulas (18) for the initial domains D^* and D. Relations (20) for the specified functions $Q_0(\zeta)$ and $Q(\zeta)$ define the conformal mappings $W = W(w)$ and $Z = Z(z)$ and the domains $\Omega^* = W(D^*)$ and $\Omega = Z(D)$.

The function $Q(\zeta)$ is found by comparing (20) and (19):

$$Q^{-1} = N_2(N_0)\omega(\zeta)\chi^{\nu-1}(\zeta)\,\mathrm{e}^{M(\zeta)}, \qquad N_2 = \mathrm{const} > 0. \tag{21}$$

We note that $\omega(\zeta)$ and $\chi(\zeta)$ do not depend on $T = (t_1, \ldots, t_{n-1})$, and $M(\zeta)$ is a function only of a fixed constant N_0.

For polygons $P \subset G$, the function $Q(\zeta)$ in (21) possesses the properties

$$\arg Q(t) = 0, \quad |t| < 1, \qquad |\ln N_2 Q(t)| \le M < \infty, \quad |t| < \infty.$$

Thus, $\arg\big(d(Z - z)/dt\big) = 0$, $|t| < 1$, and, hence, conformal mapping $Z = Z(z)$ transforms a polygon P to a polygon $Z(P)$ with parallel sides and sides lengths:

$$L_j = \int\limits_{t_{j-1}}^{t_j} |Q(t)| \left|\frac{dz}{dt}\right| dt, \qquad j = \overline{1, n-1}. \tag{22}$$

We fix the vector $T \subset R$, and thus the polygons P and $Z(P)$. The initial point $Z(-1) = 0$ of the polygon $P(Z)$ is specified, and at the end point, we set $Z(1) = 1$, which can be achieved by extending the mapping $Z(\zeta)$. We calculate the variation $\delta L_j(T)$ through the variation δT. For system (22), which corresponds to Kirchhoff flow, if condition (6) is satisfied, the equalities $\delta Z(\pm 1) = 0$ imply that $\delta Z = 0$, and, hence, $\delta T = 0$ [5, pp. 153–158]. Let us return to system (5), for which $\delta l_k = 0$ ($k = \overline{1, n-1}$) and, as shown, $\delta T = 0$, which implies that $(Dl/DT)(T) \neq 0$. Thus, we established the *topological similarity principle* for problems of parameters for the general jet flow and the Kirchhoff flow.

Theorem 3. *The solutions $T = (t_1, \ldots, t_{n-1})$ of Eq.* (5)*, corresponding to the general jet problem for a simple polygon $P \subset G$, subject to conditions* (6)*, are locally unique, i.e., $Dl/DT \neq 0$. If there exists a polygon $P^0 \subset G$ for which the solution of Eq.* (5) *is unique, it is also unique for any $P \subset G$.*

3.3. Convergent Algorithm of Numerical Solution of the Problem of Parameters. We take the segment of the straight line P^0 that connects the ends z_0 and z_{n+1} of the polygon P, fix the points z_k^0 $(k = \overline{1,n})$ on this segment, and construct the set of polygons $\{P^\nu\}$, where $\nu = (\nu_0, \dots, \nu_{n+1})$ and $\nu_k \in [0,1]$, that includes P^0 and the initial polygon $P = P^1$ [6]. We introduce the geometrical characteristic $p^\nu = (l^\nu, \alpha^\nu)$ of the polygon P^ν $[l^\nu = (l_1^{\nu_1}, \dots, l_{n-1}^{\nu_{n-1}})$ is the side length vector P^ν, $\alpha = (\alpha_0^{\nu_0}, \dots, \alpha_{n-1}^{\nu_{n-1}})$, and $\alpha_k^{\nu_k}\pi$ are the angles P^ν at the points $z_k^{\nu_k}$, $k = \overline{0, n-1}\,]$. Equation (5) is represented in the following equivalent form:

$$u = F(u, p), \qquad F_k = u_k l_k^{-1} g_k(T, \alpha), \quad k = \overline{1, n-1}. \tag{23}$$

Here $u_k = t_{k-1} - t_{k-2}$, $k = \overline{2, n}$; $p = (l, \alpha)$. We consider two polygons $(P^\lambda, P^\mu) \in \{P^\nu\}$ with the close characteristics

$$0 < |p^\lambda - p^\mu| \le q \ll 1$$

and write the following equation for the perturbations $v = u^\lambda - u^\mu$ [8]:

$$v = \Phi(u, q). \tag{24}$$

For the Kirchhoff and Ryabushinskii models, it is shown in [6] that if the condition $|Dl/DT| \ge \delta > 0$ is satisfied, there exists a fixed value of the parameter $q = q_0(\delta) > 0$ such that the perturbation operator $\Phi(u, q_0)$ is compressing on a certain set $S \subset \mathbb{R}^n$. This allows us to divide the process of finding the solutions $u = (u_1, \dots, u_{n-1})$ of Eq. (23) into a *finite number of cycles*, in each of which Eq. (24) for perturbations can be solved using simple iterations.

This convergent algorithm of solving Eq. (23) is called the cyclic iteration *method* (*algorithm*). According to the inequality $Dl/DT \ne 0$ proved in Theorem 3, the cyclic iteration method is inappropriate for the general jet problem.

Theorem 4. *Equation* (23) *for the parameters corresponding to the general jet problem can be solved using the convergent cyclic iteration algorithm.*

3.4. Optimization. In external aerodynamic problems, the drag of a polygon $P \in G(\delta)$ in flow is used as the target functional:

$$F_0(P) = \int_{-1}^{1} \Big|\frac{dw}{dz}(t)\Big|\, dt = N_0 \int_{-1}^{1} \Big|\chi^\nu \Pi\, \mathrm{e}^M\Big|^{-1} dt.$$

We set $F(P) = [F_0(P)]^{-1}$ and seek $\sup\limits_G F(P)$. The functional $F(P)$ is continuously differentiable with respect to the explicitly included parameters t_k and β_j and completely obeys the conditions of Theorem 1.

4. OPTIMIZATION IN THE CLASS OF CURVED OBSTACLES

We consider the problem studied in Sec. 3 for curved obstacles without representing them as convergent sequences of polygons. For this, we use the method employed in [9] to filtration problems.

4.1. Curved Boundary. We construct a certain Lyapunov curve $\Gamma(\mu) \subset C^{\alpha+1}$ $(\alpha > 0)$ that approximates the polygon P so that the derivative $dz/d\zeta$ of the conformal mapping $z\colon\ E \to D(\Gamma)$, $\partial D(\Gamma) = \Gamma \cup L$ can be represented explicitly in the form of (2), where $\mu > 0$ is the approximation parameter. We introduce the following notation: $t_k^\pm = t_k \pm r_k$, $r_k(\mu) = \mu \inf\{(t_k - t_{k-1}), (t_{k+1} - t_k)\}$, $k = \overline{1, n+1}$, $0 < \mu \le 1/3$, $t_0^\pm = t_0 = -1$, $t_{n+1}^- = t_{n+1} = 1$, $\Delta_k = [t_k^-, t_k^+]$, and $\Delta_k^+ = [t_k^+, t_{k+1}^-]$. We consider the function $\theta(t) = \bar\theta(t) - 1$ that satisfies the conditions $\theta(t) = \delta_k \pi$ $(t \in \Delta_k^+$ and $\theta = 0$; $|t| > 1)$:

$$\theta = [\delta_k(t - t_k^-) + \delta_{k-1}(t_k^+ - t)]|\Delta_k|^{-1} \equiv \theta_k(t), \qquad t \in \Delta_k.$$

Here $\delta_k = \bar\delta_k - 1$. The function $\theta(t, \mu)$ $(|t| < 1)$ constructed here is continuous and is uniformly bounded irrespective of the quantities $(t_{k+1} - t_k) \ge 0$ $(k = \overline{0, n})$ and $|\theta| \le \sup\limits_k |\delta_k|$.

We consider the function

$$\Pi_\theta(\zeta) = (1 - \zeta^2)^{-1/2} \exp\Big(\int_{-1}^{1} \frac{\theta(t, \mu)\, dt}{t - \zeta}\Big),$$

which is the derivative of the conformal mapping

$$Z = \int\limits_{-1}^{\zeta} \Pi_\theta(\zeta)\,d\zeta, \qquad Z\colon\ \ E \to D(\overline{\Gamma}), \quad \overline{\Gamma}(\mu) = \Gamma \cup P_0 \cup P_{n+1}.$$

In this case, the domain $D(\Gamma)$ is limited by a certain approximating curve $\Gamma(\mu)$ with slope $\pi\theta(t)$ of the tangent to the Ox axis and by the half-lines $P_0 = (z_0, \infty)$ and $P_{n+1} = (z_{n+1}, \infty)$. Calculating the Cauchy type integral in the representation of Π_θ, we obtain the following expression for the function $\Pi_\theta(\zeta)$, which is accurate to within the extension constant:

$$\Pi_\theta = (1-\zeta^2)^{-1/2} \prod_{k=0}^{n} \Big(\frac{t_k^+ - \zeta}{t_k^- - \zeta}\Big)^{\gamma_k(\zeta)} \Big(\frac{t_{k+1}^- - \zeta}{t_k^+ - \zeta}\Big)^{\delta_k}.$$

Here $\gamma_k = (a_k + b_k\zeta)|\Delta_k|^{-1}$ ($a_k = t_k^+\delta_{k-1} - \delta_k t_k^-$ and $b_k = \delta_k - \delta_{k-1}$; $k = \overline{1,n}$), $\gamma_0 = 0$, $t_0^\pm = t_0 = -1$, $t_{n+1}^- = t_{n+1} = 1$, and $|\Delta_k| = (t_k^+ - t_k^-)$ ($k = \overline{1,n}$).

By the construction, $\theta(t,\mu) \to \delta_k$ and $t \in [t_{k-1}, t_k]$ as $\mu \to 0$ and, hence, the curve $\Gamma(\mu)$ converges to the specified polygon P, and the derivative $(dz_\theta/d\zeta)(\mu)$ of the conformal mapping $z_\theta\colon E \to D(\Gamma)$ is represented as (18), in which Π_θ and M_θ should be substituted for Π and M.

Similarly to the problem for the polygon, we consider Eq. (5) for the vector $T = (t_1, \dots, t_{n-1})$, in which

$$g_k(T,\beta) = \Big| \int\limits_{t_{k-1}}^{t_k} \frac{dz_\theta}{dt}\,dt \Big| = |z_k - z_{k-1}|, \qquad k = \overline{1,n-1}. \tag{25}$$

The definition of the curve $\Gamma(\mu)$ includes the specified geometrical characteristic (l,β) [$l = (l_1, \dots, l_n)$ and $\beta = (\beta_0, \dots, \beta_{n+1})$] of the basic polygon P, with which $\Gamma(\mu)$ coincides for $\mu = 0$. It is assumed that the vector (l,β) obeys the nondegeneracy conditions (3) for the polygon P.

The conformal mapping $z_\theta\colon\ E \to D(\Gamma)$ transforms t_k ($k = \overline{0,n+1}$) to the points $z_{\theta k}(\mu) \in \Gamma(\mu)$ — the vertices of a certain polygon $P_\theta(\mu)$ approximated by the curve $\Gamma(\mu)$; the lengths of the sides $P_\theta(\mu)$ and P coincide, and the external angles $\pi\beta_k(\mu)$ and $\pi\beta_k$, generally speaking, take different values.

An arbitrary vector T substituted into (5) corresponds to a certain curve $\Gamma(\mu, T)$ approximating the polygon $P(\mu, T)$. Equations (5) are the conditions of coincidence of $\Gamma(\mu, T)$ and $\Gamma(\mu)$, and, hence, the conditions of coincidence of $P(\mu, T)$ and $P(\mu)$.

4.2. Solvability of the Problem. After the integration segments in (25) are reduced to the segment $[0,1]$ similarly to [5], it is established that the functions $g_k(T,\beta)$ are continuously differentiable with respect to t_i ($i = \overline{1,n-1}$) and for the vector T, estimates (4) are valid (see also [9]).

Since the operator g: $\mathbb{R}^n \to \mathbb{R}^{n-1}$ [$g = (g_1, \dots, g_{n-1})$] in (25) is continuously differentiable with respect to t_k on the set $T \in R$ and does not have stationary points on its boundary, then under the Schauder theorem, Eq. (25) has at least one solution. Hence the following theorem is proved.

Theorem 5 (existence theorem)**.** *Let the base polygon P be nondegenerate. Then, for the curve corresponding to it $\Gamma(\mu)$, Eq.* (25) *has at least one solution $T = (t_1, \dots, t_{n-1})$ that belongs to the set R determined in* (4).

The optimization problem is studied in the same way as in Sec. 3.

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